The reverse bra-ket method

By J.A.J. van Leunen

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Abstract

Hilbert spaces can store discrete quaternions and quaternionic continuums in the eigenspaces of operators that reside in these Hilbert spaces. The reverse bra-ket method can create natural parameter spaces from quaternionic number systems and can relate pairs of functions and their parameter spaces with eigenspaces and eigenvectors of corresponding operators that reside in non-separable Hilbert spaces. This also works for separable Hilbert spaces and the defining functions relate the separable Hilbert space with its non-separable companion. The method links Hilbert space technology with function technology, differential technology and integral technology.

1 Introduction

A need exists to be able to treat fields independent of the equations that describe their behavior. This is possible by exploiting the fact that Hilbert spaces can store discrete quaternions and quaternionic continuums in the eigenspaces of operators that reside in Hilbert spaces [1] [2] [3]. The reverse bra-ket method can create natural parameter spaces from quaternionic number systems and can relate pairs of functions and their parameter spaces with eigenspaces and eigenvectors of corresponding operators that reside in non-separable Hilbert spaces [4]. This also works for separable Hilbert spaces and the defining functions relate the separable Hilbert space with its non-separable companion. Quaternionic number systems exist in several versions that differ in their symmetry flavor. Thus, in Hilbert spaces several different versions of parameter spaces can coexist. It is possible that a parameter space floats over another parameter space. This is used by elementary objects, whose platforms float over a background space. The reverse bra-ket method is extensively applied in The Hilbert Book Test Model [8].

2 Quaternionic Hilbert spaces

Separable Hilbert spaces are linear vector spaces in which an inner product is defined. This inner product relates each pair of Hilbert vectors. The value of that inner product must be a member of a division ring. Suitable division rings are real numbers, complex numbers and quaternions. This paper uses quaternionic Hilbert spaces [2][3][4].

Paul Dirac introduced the bra-ket notation that eases the formulation of Hilbert space habits [5].

$$\langle x|y\rangle = \langle y|x\rangle^* \tag{1}$$

$$\langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle \tag{2}$$

$$\langle \alpha x|y\rangle = \alpha\,\langle x|y\rangle \tag{3}$$

$$\langle x|\alpha y\rangle = \langle x|y\rangle\,\alpha^* \tag{4}$$

$\langle x|$ is a bra vector. $|y\rangle$ is a ket vector. α is a quaternion.

This paper considers Hilbert spaces as no more and no less than structured storage media for dynamic geometrical data that have an Euclidean signature. Quaternions are ideally suited for the storage of such data. Quaternionic Hilbert spaces are described in “Quaternions and quaternionic Hilbert spaces” [6].

The operators of separable Hilbert spaces have countable eigenspaces. Each infinite dimensional separable Hilbert space owns a Gelfand triple. The Gelfand triple embeds this separable Hilbert space and offers as an extra service operators that feature continuums as eigenspaces. In the corresponding subspaces the definition of dimension loses its sense.

2.1 Representing operators and their eigenspaces by continuous functions

Operators map Hilbert vectors onto other Hilbert vectors. Via the inner product the operator T may be linked to an adjoint operator $T^\dagger$.

$$\langle Tx|y\rangle \equiv \langle x|T^\dagger y\rangle \tag{1}$$

$$\langle Tx|y\rangle = \langle y|Tx\rangle^* = \langle T^\dagger y|x\rangle^* \tag{2}$$

A linear quaternionic operator T, which owns an adjoint operator $T^\dagger$ is normal when

$$T^\dagger T = T T^\dagger \tag{3}$$

$T_0 = (T + T^\dagger)/2$ is a self adjoint operator and $\boldsymbol{T} = (T - T^\dagger)/2$ is an imaginary normal operator. Self adjoint operators are also Hermitian operators. Imaginary normal operators are also anti-Hermitian operators.

By using what we will call ***reverse bra-ket notation***, operators that reside in the Hilbert space and correspond to continuous functions, can easily be defined by starting from an orthonormal base of vectors. In this base the vectors are normalized and are mutually orthogonal. The vectors span a subspace of the Hilbert space. We will attach eigenvalues to these base vectors via the ***reverse bra-ket notation***. This works both in separable Hilbert spaces as well as in non-separable Hilbert spaces.

Let $\{q_i\}$ be the set of ***rational*** quaternions in a selected quaternionic number system and let $\{|q_i\rangle\}$ be the set of corresponding base vectors. They are eigenvectors of a normal operator $\mathcal{R}$. Here we enumerate the base vectors with index i.

$$\mathcal{R} \equiv |q_i\rangle q_i \langle q_i| = |q_i\rangle \Re(q_i) \langle q_i| \tag{4}$$

$\mathcal{R}$ is the configuration parameter space operator. $\Re(q)$ is a quaternionic function, whose target equals its parameter space.

This notation must not be interpreted as a simple outer product between a ket vector $|q_i\rangle$, a quaternion q_i and a bra vector $\langle q_i|$. It involves a complete set of eigenvalues $\{q_i\}$ and a complete orthomodular set of Hilbert vectors $\{|q_i\rangle\}$. It implies a summation over these constituents, such that for all bra's $\langle x|$ and all ket's $|y\rangle$:

$$\langle x|\mathcal{R}\, y\rangle = \sum_i \langle x|q_i\rangle q_i \langle q_i|y\rangle \tag{5}$$

$\mathcal{R}_0 = (\mathcal{R} + \mathcal{R}^\dagger)/2$ is a self-adjoint operator. Its eigenvalues can be used to arrange the order of the eigenvectors by enumerating them with the eigenvalues. The ordered eigenvalues can be interpreted as ***progression values***.

$\boldsymbol{\mathcal{R}} = (\mathcal{R} - \mathcal{R}^\dagger)/2$ is an imaginary operator. Its eigenvalues can also be used to order the eigenvectors. The eigenvalues can be interpreted as ***spatial values*** and can be ordered in several ways.

Let $f(q)$ be a mostly continuous quaternionic function. Now the reverse bra-ket notation defines operator f as:

$$f \equiv |q_i\rangle f(q_i)\langle q_i| \tag{6}$$

f defines a new operator that is based on function $f(q)$. Here we suppose that the target values of f belong to the same version of the quaternionic number system as its parameter space does.

Operator f has a countable set of discrete quaternionic eigenvalues.

For this operator the reverse bra-ket notation is a shorthand for

$$\langle x|f\, y\rangle = \sum_i \langle x|q_i\rangle f(q_i)\langle q_i|y\rangle \tag{7}$$

Alternative formulations for the reverse bra-ket definition are:

$$f \equiv |q_i\rangle f(q_i)\langle q_i| = |q_i\rangle\langle f(q_i) q_i| = |q_i\rangle\langle f q_i| = |f^*(q_i) q_i\rangle\langle q_i| = |f^\dagger q_i\rangle q_i \langle q_i| \tag{8}$$

Here we used the same symbol for the operator f and the function $f(q_i)$.

The left side of (7) only equals the right side when domain over which the summation is taken is restricted to the region of the parameter space $\mathcal{R}$ where $f(q)$ is sufficiently continuous.

2.2 Symmetry centers

We can define a category of anti-Hermitian operators $\{\mathfrak{S}_n^x\}$that have no Hermitian part and that are distinguished by the way that their eigenspace is ordered by applying a polar coordinate system. We call them symmetry centers $\mathfrak{S}_n^x$. A polar ordering always start with a selected Cartesian ordering. The geometric center of the eigenspace of the symmetry center floats on a background parameter space of the normal reference operator $\mathcal{R}$, whose eigenspace defines a full quaternionic parameter space. The eigenspace of the symmetry center $\mathfrak{S}_n^x$ acts as a three dimensional spatial parameter space. The super script ^{x} refers to the symmetry flavor of $\mathfrak{S}_n^x$. The subscript $_n$enumerates the symmetry centers. Sometimes we omit the subscript.

$$\mathfrak{S}^x = |\mathfrak{s}_i^x\rangle \mathfrak{s}_i^x \langle \mathfrak{s}_i^x| \tag{1}$$

$$\mathfrak{S}^{x\dagger} = -\mathfrak{S}^x \tag{2}$$

2.1 Continuum eigenspaces

In a non-separable Hilbert space, such as the Gelfand triple, the continuous function $\mathcal{F}(q)$ can be used to define an operator, which features a continuum eigenspace. We start with defining a continuum parameter space.

$$\mathfrak{R} = |q\rangle q\langle q| = |q\rangle \mathfrak{R}(q)\langle q| \tag{1}$$

This definition relates the separable Hilbert space and its companion Gelfand triple.

$$\mathcal{F} = |q\rangle\mathcal{F}(q)\langle q| \tag{2}$$

Via the continuous quaternionic function $\mathcal{F}(q)$, the operator $\mathcal{F}$ defines a curved continuum $\mathcal{F}$. This operator and the continuum reside in the Gelfand triple, which is a non-separable Hilbert space.

The function $\mathcal{F}(q)$ uses the eigenspace of the reference operator $\mathfrak{R}$ as a flat parameter space that is spanned by a quaternionic number system $\{q\}$. The continuum $\mathcal{F}$ represents the target space of function $\mathcal{F}(q)$.

Here we no longer enumerate the base vectors with index i. We just use the name of the parameter. If no conflict arises, then we will use the same symbol for the defining function, the defined operator and the continuum that is represented by the eigenspace.

For the shorthand of the reverse bra-ket notation of operator $\mathcal{F}$ the integral over q replaces the summation over q_i.

$$\langle x|\mathcal{F}\, y\rangle = \sum_i \langle x|q_i\rangle\mathcal{F}(q_i)\langle q_i|y\rangle \approx \int_q \langle x|q\rangle\mathcal{F}(q)\langle q|y\rangle\; dq \tag{3}$$

The integral only equals the sum if the function $\mathcal{F}(q)$ is sufficiently continuous in the domain over which the integration takes place. Otherwise the left side only equals the right side when domain is restricted to the region of the parameter space $\mathfrak{R}$ where $\mathcal{F}(q)$ is sufficiently continuous.

Remember that quaternionic number systems exist in several versions, thus also the operators f and $\mathcal{F}$ exist in these versions. The same holds for the parameter space operators. When relevant, we will use superscripts in order to differentiate between these versions.

Thus, operator $f^x = |q_i^x\rangle f^x(q_i^x)\langle q_i^x|$ is a specific version of operator f. Function $f^x(q_i^x)$ uses parameter space $\mathcal{R}^x$.

Similarly, $\mathcal{F}^x = |q^x\rangle\mathcal{F}^x(q^x)\langle q^x|$ is a specific version of operator $\mathcal{F}$. Function $\mathcal{F}^x(q^x)$ and continuum $\mathcal{F}^x$ use parameter space $\mathfrak{R}^x$. If the operator $\mathcal{F}^x$ that resides in the Gelfand triple $\mathcal{H}$ uses the same defining function as the operator $\mathcal{F}^x$ that resides in the separable Hilbert space, then both operators belong to the same quaternionic ordering version.

In general the dimension of a subspace loses its significance in the non-separable Hilbert space.

The continuums that appear as eigenspaces in the non-separable Hilbert space $\mathcal{H}$ can be considered as quaternionic functions that also have a representation in the corresponding infinite dimensional separable Hilbert space $\mathfrak{H}$. Both representations use a flat parameter space $\mathfrak{R}^x$ or $\mathcal{R}^x$ that is spanned by quaternions. $\mathcal{R}^x$ is spanned by rational quaternions.

The parameter space operators will be treated as reference operators. The rational quaternionic eigenvalues $\{q_i^x\}$ that occur as eigenvalues of the reference operator $\mathcal{R}^x$ in the separable Hilbert space map onto the rational quaternionic eigenvalues $\{q_i^x\}$ that occur as subset of the quaternionic eigenvalues $\{q^x\}$ of the reference operator $\mathfrak{R}^x$ in the Gelfand triple. In this way the reference operator $\mathcal{R}^x$ in the infinite dimensional separable Hilbert space $\mathfrak{H}$ relates directly to the reference operator $\mathfrak{R}^x$, which resides in the Gelfand triple $\mathcal{H}$.

All operators that reside in the Gelfand triple and are defined via a mostly continuous quaternionic function have a representation in the separable Hilbert space.

3 Types of operators

Only a special type of operators can directly be handled by the reverse bra-ket method. In that case the defining function must be available within the realm of the Hilbert space. All operators that are defined in the separable Hilbert space and that can be represented by a sufficiently continuous function, possess a smoothing companion in the non-separable Hilbert space. The integration process that is used by the reverse bra-ket method can handle point-like discontinuities and closed cavities in the parameter space of the defining function, where the defining function does not exist. These artifacts are handled by separating them from the ***validity domain*** [7].

Other types of operators are:

- Stochastic operators
  - These operators get their eigenvalues via mechanisms that reside outside of the realm of the Hilbert space and use stochastic processes in order to generate the eigenvalues.
- Density operators
  - If a stochastic operator generates a ***coherent swarm*** of eigenvalues that can be characterized by a continuous location density distribution, then the reverse bra-ket method can be used to define the corresponding density operator.
- Function operators
  - Function operators act on functions and in that way they produce new functions that can be used as defining functions of the corresponding operator.
- Partial differential operators
  - These are special kinds of function operators.
  - The existence of partial differentials of quaternionic functions create the existence of partial differential operators that work in combination with the operators that define the function related operator.

In quaternionic differential calculus the differential operators work as multipliers.

If $\mathfrak{D}$ is a partial differential operator and $\mathcal{G} = \mathfrak{D}\mathcal{F}$ for a category of functions $\{\mathcal{F}\}$, where $\mathcal{G}$ is sufficiently continuous, then for all bra's $\langle x|$ and all ket's $|y\rangle$ hold:

$$\langle x|\mathcal{G}\, y\rangle = \langle x|\mathfrak{D}\mathcal{F}\, y\rangle \approx \int_q \langle x|q\rangle \mathfrak{D}\mathcal{F}(q)\langle q|y\rangle\, dq = \int_q \langle x|q\rangle \mathcal{G}(q)\langle q|y\rangle\, dq \quad (1)$$

Differential operators work on the category of operators that can be represented by defining functions, which can be differentiated. Especially the Hermitian kind of these operators appear to be of interest for application in physical theories.

Some Hermitian partial differential operators do not mix scalar and vector parts of functions. These are:

∇_0

$\nabla_0\nabla_0$

$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle$

These operators can be combined in additions as well as in products. Two particular operators are:

$\nabla\nabla^{*} = \nabla^{*}\nabla = \nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle$

$\mathfrak{D} = -\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle$

The last one is the quaternionic version of d'Alembert's operator. The first one can be split into ∇ and ∇^{*}. The second one cannot be split into quaternionic first order partial differential operators.

The field $\mathfrak{F}$ is considered to be regular in spatial regions where the defining function $\mathfrak{F}(q)$ obeys

$$\langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\mathfrak{F} = 0 \tag{2}$$

Similar considerations hold for regions where:

$$\nabla\nabla^{*}\mathfrak{F} = (\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\mathfrak{F} = 0 \tag{3}$$

$$\mathfrak{D}\mathfrak{F} = (-\nabla_0\nabla_0 + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\mathfrak{F} = 0 \tag{4}$$

References

[1] In 1843 quaternions were discovered by Rowan Hamilton. http://en.wikipedia.org/wiki/History_of_quaternions

[2] Quantum logic was introduced by Garret Birkhoff and John von Neumann in their paper: G. Birkhoff and J. von Neumann, (1936) The Logic of Quantum Mechanics, *Annals of Mathematics*, **37**, 823–843

This paper also indicates the relation between this orthomodular lattice and separable Hilbert spaces.

[3] The Hilbert space was discovered in the first decades of the 20-th century by David Hilbert and others. See http://en.wikipedia.org/wiki/Hilbert_space.

[4] In the sixties Israel Gelfand and Georgyi Shilov introduced a way to model continuums via an extension of the separable Hilbert space into a so called Gelfand triple. The Gelfand triple often gets the name rigged Hilbert space. It is a non-separable Hilbert space. http://www.encyclopediaofmath.org/index.php?title=Rigged_Hilbert_space .

[5] Paul Dirac introduced the bra-ket notation, which popularized the usage of Hilbert spaces. Dirac also introduced its delta function, which is a generalized function. Spaces of generalized functions offered continuums before the Gelfand triple arrived.

See: P.A.M. Dirac.(1958), The Principles of Quantum Mechanics, Fourth edition, Oxford Univerity Press, ISBN 978 0 19 852011 5.

[6] Quaternionic function theory and quaternionic Hilbert spaces are treated in: J.A.J. van Leunen. (2015) Quaternions and Hilbert spaces. http://vixra.org/abs/1411.0178 .

[7] The Generalized Stokes Theorem" treats integration of quaternionic manifolds. http://vixra.org/abs/1512.0340 .

[8] The Hilbert Book Test Model. http://vixra.org/abs/1603.0021 .