Relational Structures

By J.A.J. van Leunen

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Abstract

Lattices form the foundation of important theories. One is classical logic. Another is quantum logic, which is an orthomodular lattice.

1 Lattices

A lattice is a set of elements $a,b,c,...$ that is closed for the connections $\cap$ and $\cup$. These connections obey:

- The set is ***partially ordered***.
  - This means that with each pair of elements a,b belongs an element c, such that $a \subset c$ and $b \subset c$.
- The set is a $\cap$ ***half lattice.***
  - This means that with each pair of elements a,b an element c exists, such that $c = a \cap b$.
- The set is a $\cup$ half lattice.
  - This means that with each pair of elements a,b an element c exists, such that $c = a \cup b$.
- The set is a lattice.
  - This means that the set is both a $\cap$ half lattice and a $\cup$ half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a$$
$$(a \cap b) \cap c = a \cap (b \cap c)$$
$$a \cap (a \cup b) = a$$

$$a \cup b = b \cup a$$
$$(a \cup b) \cup c = a \cup (b \cup c)$$
$$a \cup (a \cap b) = a$$

The lattice has a ***partial order inclusion*** $\subset$:

$$a \subset b \Leftrightarrow a \cap b = a$$

A ***complementary lattice*** contains two elements n and e and with each element *a* it contains a complementary element $a^{'}$

such that:

$$a \cap a^{'} = n$$
$$a \cap n = n$$
$$a \cap e = a$$
$$a \cup a^{'} = e$$
$$a \cup e = e$$
$$a \cup n = a$$

An ***orthocomplemented lattice*** contains two elements n and e and with each element *a* it contains an element a'' such that:

$$a \cup a^{''} = e$$
$$a \cap a^{''} = n$$
$$\left(a^{''}\right)^{''} = a$$
$$a \subset b \Leftrightarrow b^{''} \subset a^{''}$$

e is the ***unity element***; n is the ***null element*** of the lattice

A ***distributive lattice*** supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$
$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

A ***modular lattice*** supports:

$$(a \cap b) \cup (a \cap c) = a \cap \big(b \cup (a \cap c)\big)$$

A ***weak modular lattice*** supports instead:

There exists an element d such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c)$$

where d obeys:

$$(a \cup b) \cap d = d$$
$$a \cap d = n$$
$$b \cap d = n$$
$$(a \subset g) and (b \subset g) \Leftrightarrow d \subset g$$

In an ***atomic lattice*** holds

$$\exists \{p \ni L\} \forall \{x \ni L\} \{x \subset p \Rightarrow x = n\}$$
$$\forall \{a \ni L\} \forall \{x \ni L\} \{(a \subseteq x \subseteq (a \cap p) \Rightarrow [(x = a) or (x = a \cap p)])\}$$

p is an atom

2 Well known lattices

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.

Quantum logic has the structure of an orthocomplemented weakly modular and atomic lattice.

It is also called an ***orthomodular lattice***.

Both lattices are atomic lattices.